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Recurrent Versus Diffusive Dynamics for a Kicked Quantum System

M. Combescure[1]

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We study the dynamics of a two-level quantum system subject to a time-dependent kicking perturbation modulated along the Thue–Morse sequence. For a nontrivial set of the parameters, the quantum autocorrelation function is explicitly calculated, and splits into a purely recurrent and a purely diffusive part. Furthermore, the diffusive part is directly related to the (singular continuous) correlation measure of the Thue–Morse sequence.

KEY WORDS: Quantum instability; Thue–Morse sequence; quantum dynamics; two-level quantum systems.

1. INTRODUCTION

The dynamics of periodically driven quantum systems has received a great deal of attention in recent years, especially regarding their long-time behavior: is it regular and recurrent, like the classical dynamics of regular systems, or is it irregular and diffusive, as for classically chaotic systems?[5–8,12–16,23,27–29] The prototype of these investigations is the well-known "kicked rotator," which has been intensively studied, both on the classical and the quantum level: beyond a critical value of the size of the "kicks," the classical dynamics is entirely chaotic, whereas the quantum dynamics exhibits, at least numerically, the so-called "quantum suppression of classical chaos."[7,13] For time-periodic systems, a convenient mathematical tool for the study of the quantum long-time behavior is the Floquet operator, namely the quantum evolution operator $U(T, 0)$ between times 0 and T (T is the time period); more precisely, the spectral properties of $U(T, 0)$ contain the main information on the quantum dynamics: to the

[1] Laboratoire de Physique Théorique (Laboratoire Associée au CNRS), bât. 211, 91405 Orsay, France.

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point spectral subspace of $U(T, 0)$ corresponds a recurrent quantum evolution, whereas the continuous spectral subspace of $U(T, 0)$ yields a diffusive quantum evolution, with suitably decaying autocorrelation functions, as a result of the Riemann–Lebesgue lemma.

With this characterization in mind, it is natural to expect that the "quantum suppression of classical chaos" is the persistence of a pure-point character of the spectrum of $U(T, 0)$ in the large coupling limit. Several attempts in this direction have not yet provided a completely satisfactory answer.[6,7] Many other time-periodic quantum systems have been investigated, either rigorously[3,9-11] or numerically[7,8,12] (also see references therein quoted). Most of them indicate that time-periodic quantum systems are much more stable than their classical analogs, even in the large coupling limit. However, it is expected that some randomness in the time-dependent driving force destroys this stability of the quantum evolution: this has been actually exhibited by Guarneri[15] for a "randomly kicked quantum rotator," where the gaps between two consecutive kicks are independent nonnegative random variables distributed according to a common law. Here, of course, one has no Floquet analysis at one's disposal to study the long-time quantum evolution, and the occurrence of an instability in the quantum dynamics is related to a diffusive growth of the averaged expectation value of the energy.

However, pure randomness in the time-dependent driving potential is not very easy to handle in general; halfway between the purely random and the purely periodic cases is the case of "deterministic disorder" induced by suitable substitution sequences. For example Luck *et al.*[21] (see also ref. 29) have considered the quantum evolution problem for two-level quantum systems with a driving perturbation which is quasiperiodic in time: the discrete kicks are generated by a Fibonacci sequence which is known to be quasiperiodic. Their numerical approach aims at the characterization of the response in the quantum evolution by analyzing the power spectrum and various correlation functions of the solution. They provide analytical and numerical evidence that the evolution exhibits in general some intermediate kind of behavior between quasiperiodic and random. Therefore, the quantum interference effects which are known to enforce stability in the time-periodic case are destroyed by simple controlled disorder such as that given by the quasiperiodic Fibonacci sequence.

In this paper, we extend this approach of the dynamics of quantum two-level systems driven by time-dependent aperiodic perturbations to another type of substitution binary sequences: we take as a prototype the well-known Thue–Morse sequence,[26] which is known to have a purely singular continuous Fourier spectrum.

Thus, $(\gamma_n)_{n\in\mathbb{Z}}$ is a doubly infinite sequence of 0 and 1, such that

$(\gamma_n)_{n\in\mathbb{N}}$ is the Thue–Morse sequence starting from $\gamma_1=1$, and $\gamma_{1-n}=\gamma_n$ ($n\in\mathbb{N}$). The time-dependent Hamiltonian is taken as

$$H(t)=E\sigma_z+\lambda\sigma_x\sum_{-\infty}^{+\infty}\gamma_n\,\delta(t-n) \tag{1.1}$$

where E and λ are real numbers, and σ_x, σ_z the Pauli matrices

$$\sigma_x=\begin{pmatrix}0&1\\1&0\end{pmatrix},\qquad \sigma_z=\begin{pmatrix}1&0\\0&-1\end{pmatrix} \tag{1.2}$$

Thus, a kick occurs at time n if and only if $\gamma_n=1$. To (1.1) can be associated a unitary time evolution operator $U(t,s)$ satisfying the time-dependent Schrödinger equation. We shall prove the following results:

(i) There exists a nontrivial set $\tilde{\mathscr{E}}$ of parameters E, λ (that seems to be dense in the square $[0,\pi]^2$) such that $U(2^n,0)=\mathbb{1}$ for some $n\in\mathbb{N}$.

(ii) Under condition (i), the quantum dynamics is both recurrent and diffusive. Given any initial state, the quantum autocorrelation function (in time) is the Fourier transform of a measure which is the sum of a pure point measure and of a singular continuous measure. The latter is the suitably scaled "Riesz measure" arising from the Fourier transform of the Thue–Morse sequence.[26]

This descriptive formulation of the results will be given a precise mathematical form in Section 2 (Theorems 1 and 2). In Section 3, we describe precisely the set $\tilde{\mathscr{E}}$ of values of the parameters (E,λ) such that (i) and (ii) hold. We believe that this approach can be generalized to other types of substitution sequences, either binary or not, even when the substitution is of nonconstant length. This will be done in another publication.

We conclude this introduction by stressing the close relationship between our time-dependent problem and the transfer matrix problem in quasicrystals or aperiodic crystals which has been dealt with recently in many papers. For the Thue–Morse case, see in particular refs. 2, 4, 22, and 30 and references therein contained.

2. THE QUANTUM DYNAMICS FOR $H(t)=E\sigma_z+\lambda\sigma_x\sum_{-\infty}^{+\infty}\gamma_n\,\delta(t-n)$

$(\gamma_n)_{\mathbb{Z}}$ is a sequence of 0 and 1 given by the Thue–Morse substitution rule

$$\begin{pmatrix}0\\1\end{pmatrix}\xrightarrow{\zeta}\begin{pmatrix}10\\01\end{pmatrix} \tag{2.1}$$

where we have chosen, by convention, to write words from the right to the left (the reason will appear clearly below). The successive substitution rule is defined inductively in the following way:

$$\zeta^n\begin{pmatrix}0\\1\end{pmatrix}=\begin{pmatrix}\zeta^{n-1}(1)\,\zeta^{n-1}(0)\\\zeta^{n-1}(0)\,\zeta^{n-1}(1)\end{pmatrix}\tag{2.2}$$

From (2.1), (2.2) it is easy to see that $\zeta^n(0)$ and $\zeta^n(1)$ are words made of 2^{n-1} times the letter 0 and 2^{n-1} times the letter 1 in an inductively defined order. For example, $\zeta(1)=01$, $\zeta^2(1)=\zeta(0)\,\zeta(1)=1001$, $\zeta^3(1)=01101001$, and so on, which defines inductively the 2^n first terms of the Thue–Morse sequence (written from the right to the left).

Definition 1. Given M a finite word made of two letters a and b, let $\bar{M}$ be the word obtained from M by replacing a by b and vice versa.

Lemma 1. $\zeta^n(0)=\overline{\zeta^n(1)}$ (the proof is easily checked by induction).

Definition 2. For $n\geq 1$, let γ_n be the nth letter of the infinite sequence $\zeta^\infty(1)$ (hence $\gamma_1=1$), and let $\gamma_{1-n}=\gamma_n$.

Lemma 2. (Self-similarity of the Thue–Morse sequence.) Denote $M_n=\zeta^n(1)$; then, for any n, the infinite sequence obtained from $\zeta^\infty(1)$ by replacing each 1 by M_n and each 0 by $\bar{M}_n$ is nothing but $\zeta^\infty(1)$.

We now define the quantum dynamics. The Hilbert space of quantum states is

$$\mathscr{H}=\{(z_1,z_2)\in\mathbb{C}^2\}\tag{2.3}$$

Either by using C^∞ approximants of the δ-pulses, or by going to Howland's formalism for time-dependent Hamiltonians,[17,25] one concludes the existence of a two-parameter family of unitary operators $U(t,s)$ in $\mathscr{H}$ such that

$$\begin{aligned}U(t,s)&=e^{-iE\sigma_z(t-s)} &&\text{if}\quad n<s<t<n+1\\ \lim_{t\nearrow n+1,\,s\nearrow n}U(t,s)&=e^{-iE\sigma_z}e^{-i\lambda\sigma_x} &&\text{if}\quad \gamma_n=1\\ &=e^{-iE\sigma_z} &&\text{if}\quad \gamma_n=0\end{aligned}\tag{2.4}$$

In other words, $U(t,s)$ is discontinuous at times n such that $\gamma_n=1$ and

$$\lim_{s\nearrow n,\,t\searrow n}U(t,s)=e^{-i\lambda\sigma_x}\tag{2.5}$$

The family $\{U(t,s)\}$ solves the quantum evolution problem for the Hamiltonian (1.1) in the following sense: it is the limit of the evolution operator $U_\varepsilon(t, s)$ solving the time-dependent Schrödinger equation when each δ peak is replaced by some C^∞ approximation η_ε suitably converging to δ when ε goes to zero.

Now restricting ourselves to the discrete times n, we denote

$$U_n = \lim_{s \searrow n \quad t, t \searrow n} U(t, s) = U(n^+, n-1^+) \quad \text{(notation)} \tag{2.6}$$

so that, by the chain rule,

$$U(n^+, 0^+) = U_n U_{n-1} \cdots U_1 \tag{2.7}$$

From (2.4), we see that each U_n can only be one of the two unitary matrices

$$\begin{aligned} U &= e^{-iE\sigma_z} \\ V &= e^{-iE\sigma_z} e^{-i\lambda\sigma_x} \end{aligned} \tag{2.8}$$

depending upon whether γ_n equals 0 or 1. We immediately see a formal analogy with the transfer matrix problem for the tight-binding Schrödinger equation with controlled disorder given by the sequence γ_n.[2,4,30] Here, however, the elements of the matrix products are unitary 2×2 matrices of determinant 1, and therefore so is $U(n^+, 0^+)$ for any n.

Definition 3. Let $V(t, s)$ be the quantum evolution operator for the Hamiltonian (1.1) with γ_n replaced by $(1-\gamma_n)$. Let $\mathcal{M}_n = U(2^{n+}, 0^+)$ and $\bar{\mathcal{M}}_n = V(2^{n+}, 0^+)$.

Then from the substitution rule (2.2) and Lemma 2.1 we immediately have the following result.

Lemma 3. (i) $\mathcal{M}_{n+1} = \bar{\mathcal{M}}_n \mathcal{M}_n$

(ii) $\mathcal{M}_n = U_{2^n} U_{2^n - 1} \cdots U_1$

$\bar{\mathcal{M}}_n = V_{2^n} V_{2^n - 1} \cdots V_1$

where

$$\begin{aligned} U_n &= V\gamma_n + U(1-\gamma_n) \\ V_n &= U\gamma_n + V(1-\gamma_n) \end{aligned} \tag{2.9}$$

As in the transfer matrix approach of refs. 2, 4, and 30, the existence of a "trace map" is a central ingredient (see also ref. 1):

Lemma 4. Let $x_n = \frac{1}{2} \operatorname{tr} \mathcal{M}_n$. Then we have:

(i) $\frac{1}{2} \operatorname{tr} \bar{\mathcal{M}}_n = x_n$

(ii) $\forall n \geqslant 2$: $\quad x_{n+1} - 1 = 4x_{n-1}^2 (x_n - 1)$.

Proof. Part (i) follows easily by induction, using the invariance of the trace under permutation, and Lemma 3(i).

Part (ii) is an immediate consequence of the Cayley–Hamilton theorem for $\mathcal{M}_n$ and $\bar{\mathcal{M}}_n$ (refs. 1 and 2): Using Lemma 3(i) twice, we get

$$\mathcal{M}_{n+1} = \mathcal{M}_{n-1}\bar{\mathcal{M}}^2_{n-1}\mathcal{M}_{n-1}$$

Taking the trace (and using its cyclicity), we get

$$x_{n+1} = \tfrac{1}{2}\operatorname{tr} \bar{\mathcal{M}}^2_{n-1}\mathcal{M}^2_{n-1}$$

But $\mathcal{M}_n$ and $\bar{\mathcal{M}}_n$ are both solutions of their characteristic equation

$$X^2 - 2x_n X + \mathbb{1} = 0$$

and therefore

$$\begin{aligned} x_{n+1} &= \tfrac{1}{2}\operatorname{tr}\{(2x_{n-1}\bar{\mathcal{M}}_{n-1} - \mathbb{1})(2x_{n-1}\mathcal{M}_{n-1} - \mathbb{1})\} \\ &= 2x^2_{n-1}\operatorname{tr}\bar{\mathcal{M}}_{n-1}\mathcal{M}_{n-1} - x_{n+1}(\operatorname{tr}\mathcal{M}_{n-1} + \operatorname{tr}\bar{\mathcal{M}}_{n-1}) + 1 \\ &= 1 + 4x^2_{n-1}(x_n - 1) \end{aligned}$$

Now, starting from any given initial state Ψ_0, we denote

$$\begin{aligned} \Psi_n &= U_n U_{n-1} \cdots U_1 \Psi_0 = U(n^+, 0^+)\,\Psi_0 \\ \Phi_n &= V_n V_{n-1} \cdots V_1 \Psi_0 = V(n^+, 0^+)\,\Psi_0 \end{aligned} \tag{2.10}$$

which, using (2.7), means that Ψ_n is the quantum state just after the nth kick. Then we show that the condition $x_n = 1$ (some integer n) is a strong recurrence condition:

Theorem 1. If there is some $n \in \mathbb{N}$ such that $x_n = 1$, then:

(i) $\mathcal{M}_n = \bar{\mathcal{M}}_n = \mathbb{1}$

(ii) if $m = (p-1)\,2^n + q$, $p \in \mathbb{N}$, $0 \leqslant q \leqslant 2^n - 1$, then

$$\begin{aligned} \Psi_m &= \gamma_p \Psi_q + (1 - \gamma_p)\,\Phi_q \\ \Phi_m &= \gamma_p \Phi_q + (1 - \gamma_p)\,\Psi_q \end{aligned} \tag{2.11}$$

Proof. Part (i) results immediately from the fact that any unitary 2×2 matrix of determinant 1 is of the form $A = \mathbb{1}\cos\alpha + i\sin\alpha\,\hat{r}\cdot\sigma$, where $\hat{r} \in \mathbb{R}^3$ is unitary, α is real ($\alpha \in [0, 2\pi)$), and $\sigma = (\sigma_x, \sigma_y, \sigma_z)$ are the Pauli matrices. Therefore $\frac{1}{2}\operatorname{tr} A = \cos\alpha$ equals 1 if and only if $\alpha = 0$, i.e., $A = \mathbb{1}$.

For (ii), recall the self-similarity of the Thue–Morse sequence (Lemma 2). It implies

$$U(m^+, 0^+) = \begin{cases} U(q^+, 0^+) \cdots \underbrace{\mathscr{M}_n \bar{\mathscr{M}}_n^2 \mathscr{M}_n}_{p-1 \text{ factors}} & \text{if} \quad \gamma_p = 1 \\ V(q^+, 0^+) \cdots \underbrace{\mathscr{M}_n \bar{\mathscr{M}}_n^2 \mathscr{M}_n}_{p-1 \text{ factors}} & \text{if} \quad \gamma_p = 0 \end{cases}$$

and similarly for $V(m^+, 0^+)$. Then, due to (i), we get

$$U(m^+, 0^+) = \gamma_p U(q^+, 0^+) + (1 - \gamma_p) V(q^+, 0^+)$$
$$V(m^+, 0^+) = \gamma_p V(q^+, 0^+) + (1 - \gamma_p) U(q^+, 0^+)$$

which, applied to the initial state Ψ_0, yields the result.

We now study the quantum autocorrelation function. The spectral study of sequences with the aid of the correlation measure is a standard strategy in the study of dynamical systems.[26] We adapt this approach to our quantum evolution problem, which will appear as a powerful strategy. We show that under the "recurrent" assumption of Theorem 1, the Fourier spectrum of the substitution Thue–Morse sequence exactly manifests itself in the quantum autocorrelation measure.

We first recall the useful notions of Fourier–Bohr spectrum and correlation measure for a sequence $(m_n)_{n \in \mathbb{N}}$.[26]

Definition 4. The Fourier–Bohr spectrum of the sequence $(m_n)_{n \in \mathbb{N}}$ is the subset of values of λ in $[0, 1]$ such that

$$\lim_{N \to \infty} N^{-1} \sum_{0}^{N} m_n e^{2i\pi\lambda n} \neq 0 \tag{2.12}$$

The correlation function for the sequence $(m_n)_{n \in \mathbb{N}}$ is the following limit (when it exists):

$$C_n(m) = \lim_{N \to \infty} N^{-1} \sum_{p=0}^{N-1} \bar{m}_p m_{n+p} \tag{2.13}$$

If it exists, it is the Fourier transform of a positive measure $\sigma_m(\lambda)\, d\lambda$ on $[0, 1]$ called the correlation measure, where $\sigma_m(\lambda)$ is the weak-star limit of

$$N^{-1} \left| \sum_{p=0}^{N-1} m_p e^{2i\pi p\lambda} \right|^2 \tag{2.14}$$

Moreover, the Fourier–Bohr spectrum of the sequence $(m_p)_{p \in \mathbb{N}}$ is the support of the pure point part of the measure $\sigma_m(\lambda)$.

Remark. For suitable sequences $(m_p)_{p\in\mathbb{N}}$ defined by substitution rules, formulas (2.12), (2.14) are meaningful.[26] For the Thue–Morse sequence of $+1$ and -1 which obeys

$$\begin{aligned} m_{2p} &= (-1)^p m_p \\ m_{2p+1} &= -m_p \\ m_0 &= 1 \end{aligned} \tag{2.15}$$

the correlation measure $\sigma_m(\lambda)$ is the (weak-star) limit of the Riesz polynomials

$$\sigma_m(\lambda) = \lim_{N\to\infty} 2^{-N}(1-\cos 2\pi\lambda)(1-\cos 4\pi\lambda)\cdots(1-\cos 2^{N+1}\pi\lambda) \tag{2.16}$$

which defines a purely singular continuous measure on $[0, 1]$ (see ref. 26) and the Fourier–Bohr spectrum is empty. Note that (2.15) is one among the various possible definitions of the Thue–Morse sequence, and that $m_n = 2\gamma_{n+1} - 1$.

Let now the sequence $(m_p)_{p\in\mathbb{N}}$ be such that the limits (2.13), (2.14) exist, and assume in addition that it is of mean 0, i.e.,

$$\lim_{N\to\infty} N^{-1} \sum_0^{N-1} m_p = 0 \tag{2.17}$$

Then given any $n\in\mathbb{N}$, we take two *finite* sequences of 2^n complex numbers $(\alpha_0, \alpha_1,..., \alpha_{2^n-1})$ and $(\alpha'_0, \alpha'_1,..., \alpha'_{2^n-1})$, and we define a new sequence $(a_p)_{p\in\mathbb{N}}$ in the following way:

$$a_{p2^n+q} = \alpha_q + \alpha'_q m_p, \qquad q = 0, 1,..., 2^n - 1 \tag{2.18}$$

We then have the following result:

Lemma 5. The correlation function of the sequence $(a_p)_{p\in\mathbb{N}}$ exists and is the Fourier transform of the correlation measure

$$\sigma_a(\lambda) = \sum_{q=0}^{2^n-1} \gamma_q \delta(\lambda - q2^{-n}) + \sigma_m(2^n\lambda) \sum_{q=-2^n+1}^{2^n-1} \gamma'_q e^{2i\pi q\lambda} \tag{2.19}$$

where γ_q and γ'_q are quadratic forms in the α_q and α'_q, respectively, and $\gamma'_{-q} = \overline{\gamma'_q}$.

Proof. We calculate the correlation function $p\in\mathbb{N}\to c_p(a)$ for the sequence $(a_p)_{p\in\mathbb{N}}$ using (2.13) and (2.18); it is easy to see that it exists, and that no cross-terms $\bar{\alpha}_q\alpha'_{q'}$ contribute, since the sequence m_p is assumed to be of mean zero [see (2.17)]. This yields, for any $p \in \mathbb{N}$, and $q = 0, 1, \ldots 2^n - 1$,

$$C_{2^n p + q}(a) = 2^{-n} \sum_{q'=0}^{2^n - q - 1} [\overline{\alpha_{q'}} \alpha_{q+q'} + \overline{\alpha'_{q'}} \alpha'_{q+q'} C_p(m)] + 2^{-n} \sum_{q'=2^n - q}^{2^n - 1} [\overline{\alpha_{q'}} \alpha_{q+q'-2^n} + \overline{\alpha'_{q'}} \alpha'_{q+q'-2^n} C_{p+1}(m)] \tag{2.20}$$

Now, writing that $C_p(a)$ is the Fourier transform

$$C_p(a) = \int_0^1 e^{2i\pi\lambda p} \sigma_a(\lambda)\, d\lambda \tag{2.21}$$

we obtain the result, by noting that

$$\int_0^1 d\lambda\, e^{2i\pi\lambda(p2^n + q)} \sigma_m(2^n \lambda) = \begin{cases} C_p(m) & \text{if } q = 0 \\ 0 & \text{otherwise} \end{cases}$$

(this last equality follows by cutting the integration interval [0, 1] into 2^n subintervals and by using the 1-periodicity of $\sigma_m(\lambda)$).

Now define the quantum autocorrelation function, which, under the recurrence condition of Theorem 1, can be shown to exist and to have nice properties:

Definition 5. Given an arbitrary initial state $\Psi_0 \in \mathscr{H}$ normalized to unity and the corresponding quantum state Ψ_n at time n given by (2.10), we define $n \to C_{\Psi_0}(n)$:

$$C_{\Psi_0}(n) = \lim_{N \to \infty} N^{-1} \sum_{p=0}^{N-1} \langle \Psi_p, \Psi_{n+p} \rangle \tag{2.22}$$

when it exists, $\langle\ ,\ \rangle$ being the usual scalar product in $\mathscr{H}$.

We then have:

Theorem 2. Under the condition of Theorem 1, the quantum autocorrelation function $n \in \mathbb{Z} \to C_{\Psi_0}(n)$ defined by (2.22) exists for any initial state Ψ_0. Furthermore, it is the Fourier transform of a correlation measure $\sigma(\lambda)$ of the form (2.19).

Proof. Write

$$\Psi_n = \begin{pmatrix} a_n \\ b_n \end{pmatrix}$$

Then obviously

$$C_{\Psi_0}(n) = C_a(n) + C_b(n) \tag{2.23}$$

But using (2.11) and the fact that $\gamma_{p+1} = \frac{1}{2}(1 + m_p)$, with m_p satisfying (2.15), we get, for $m = p2^n + q$, $0 \leqslant q \leqslant 2^n - 1$,

$$\Psi_m = \frac{1}{2}(\Psi_q + \Phi_q) + \frac{m_p}{2}(\Psi_q - \Phi_q)$$

so that, if

$$\Psi_q + \Phi_q = \begin{pmatrix} \alpha_q \\ \beta_q \end{pmatrix}, \qquad \Psi_q - \Phi_q = \begin{pmatrix} \alpha'_q \\ \beta'_q \end{pmatrix}$$

we get

$$\begin{aligned} a_{p2^n+q} &= \alpha_q + m_p \alpha'_q \\ b_{p2^n+q} &= \beta_q + m_p \beta'_q \end{aligned} \qquad q = 0, 1, ..., 2^n - 1, \quad p \in \mathbb{N}$$

which is precisely the property (2.18) under which the conclusion of Lemma 5 holds.

Remark. Theorem 2 establishes that, up to a pure point part which reflects the "recurrent character" of the condition $x_n = 1$, the Fourier spectrum of the Thue–Morse sequence exactly manifests itself in the quantum autocorrelation measure. But the only condition under which Lemma 5 holds true is the mean-zero character of the Thue–Morse sequence (m_n) and the existence of its correlation measure. Therefore this approach can be generalized to other substitution sequences provided that the property

$$\exists n \in \mathbb{N}: \quad x_n = 1 \tag{2.24}$$

holds true for a nontrivial set of parameters. This will be analyzed in another work.

3. FOR WHICH SET OF PARAMETERS (E, λ) ARE THEOREMS 1 AND 2 SATISFIED?

We now want to show that property (2.24) for the Thue–Morse sequence defines a nontrivial set of parameters (E, λ) for the quantum problem defined by Hamiltonian (1). This is done by studying the trace map of Lemma 4(ii).

Defining

$$y_n = 1 - x_{n+1} \tag{3.1}$$

we see that

$$\begin{pmatrix} x_n \\ y_n \end{pmatrix} \xrightarrow{\Phi} \begin{pmatrix} x_{n+1} = 1 - y_n \\ y_{n+1} = 4x_n^2 y_n \end{pmatrix} \tag{3.2}$$

starting with initial data

$$\begin{aligned} x_1 &= \cos 2E \cos \lambda \\ y_1 &= 2 \cos^2 E \,[1 + \cos^2 \lambda \,(3 - 4 \cos^2 E)] \end{aligned} \tag{3.3}$$

It is clear that $x_n \in [-1, 1]$ and $y_n \in [0, 2]$ $\forall n \in \mathbb{N}$. Furthermore, one can check that the initial data (3.3) lie in the interior of the parabola ($\mathscr{P}$)

$$y = 2(1 - x^2) \tag{3.4}$$

Therefore, when the parameters (E, λ) vary in the square $[0, \pi]^2$, the initial data (3.3) describe the interior of the domain $\mathscr{D}$ in $\mathbb{R}^2$ defined by

$$0 \leqslant y \leqslant 2(1 - x^2) \tag{3.5}$$

Now our problem is very similar to that studied by Axel and Peyrière,[2] namely that of finding the preimages by the mapping (3.2) of the sets $\{x_n = 1, \text{some } n\}$, with the only difference that their domain of initial data (x_1, y_1) is just the complement in $\mathbb{R}^2$ of our domain $\mathscr{D}$. Thus, let for any $n \in \mathbb{N}$, $n \geqslant 2$,

$$\mathscr{E}_n = \{(x_1, y_1) \in \mathscr{D}: x_n = 1\} \tag{3.6}$$

and

$$\mathscr{E} = \bigcup_{n \geqslant 3} (\mathscr{E}_n \backslash \mathscr{E}_{n-1}) \cup \mathscr{E}_2 \tag{3.7}$$

Using the description of ref. 2, one sees easily that the sets $\mathscr{E}_n \backslash \mathscr{E}_{n-1}$ are disjoint, $n \geqslant 3$, and that

$$\begin{aligned} \mathscr{E}_2 &= \{y_1 = 0, |x_1| \leqslant 1\} \\ \mathscr{E}_3 \backslash \mathscr{E}_2 &= \{x_1 = 0, 0 \leqslant y_1 \leqslant 2\} \\ \mathscr{E}_4 \backslash \mathscr{E}_3 &= \{y_1 = 1, |x_1| \leqslant 1/\sqrt{2}\} \end{aligned}$$

See Fig. 1, where we have drawn the curves $\mathscr{E}_n$ up to $n = 7$.

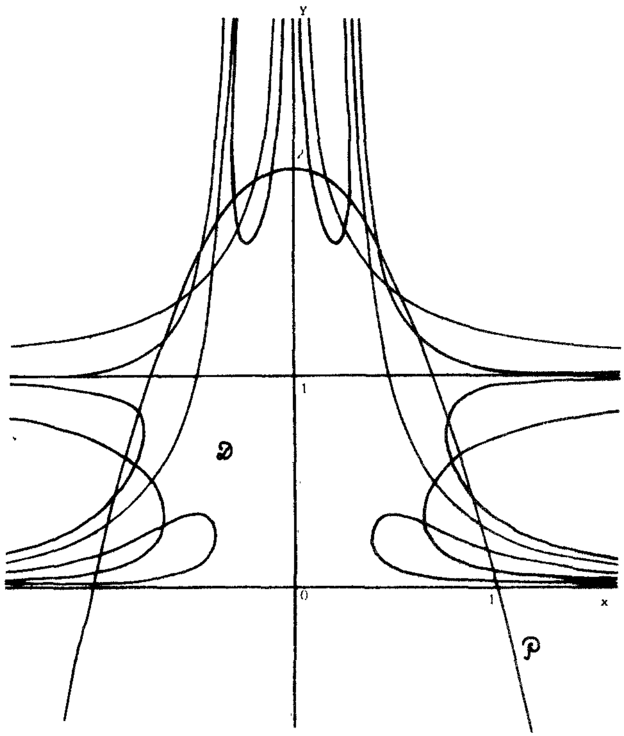

Fig. 1. The domain $\mathscr{D}$ of initial data in the (x, y) plane, and the curves $\mathscr{E}_n$ up to $n = 7$.

Moreover, for $n \geqslant 4$, $\mathscr{E}_n \backslash \mathscr{E}_{n-1}$ lies on the union of 2^{n-4} curves, so that $\mathscr{E}_n$ intersects the parabola $(\mathscr{P})$ at the points

$$x = \cos t_{n,p}$$

$$y = 2 \sin^2 t_{n,p}$$

where $t_{n,p} = \pi p 2^{-n+2}$, $p = 0, 1, \ldots, 2^{n-2}$. Furthermore, numerical computations[24,30] show that the curves $\mathscr{E}_n \backslash \mathscr{E}_{n-1}$ enter the domain $\mathscr{D}$ in a complicated way, and invade it as n becomes higher and higher. This suggests, although we have not been able to prove it, that the set $\mathscr{E}$ densely fills the domain $\mathscr{D}$, namely that any arbitrary small neighborhood of any point of $\mathscr{D}$ contains at least one element of $\mathscr{E}$ (see Fig. 1 of ref. 30). The conclusion is therefore the following:

Theorem 3. If the parameters (E, λ) are such that the initial data (3.3) of the trace mapping Φ belong to the set $\mathscr{E}$ defined by (3.7), then the conclusion of Theorem 2 holds true.

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REFERENCES

1. J. P. Allouche and J. Peyrière, *C. R. Acad. Sci. Paris* **302**:1135 (1986).
2. F. Axel and J. Peyrière, *J. Stat. Phys.* **57**:1013 (1989).
3. J. Bellissard, in *Lecture Notes in Mathematical Physics*, No. 1049 (Springer, 1985).
4. J. Bellissard, in *Théorie des Nombres et Physique* (Springer Proceedings in Physics, Vol. 47), J. M. Luck, P. Moussa, and M. Waldschmidt, eds. (Springer, Berlin, 1990).
5. R. Blümel and U. Smilansky, *Phys. Rev. Lett.* **58**:2531 (1987).
6. G. Casati and I. Guarneri, *Commun. Math. Phys.* **95**:12 (1984).
7. G. Casati, B. V. Chirikov, D. L. Shepelyanski, and I. Guarneri, *Phys. Rep.* **154**:77 (1987).
8. A. Cohen and S. Fishman, *Int. J. Mod. Phys. B* **2**:103 (1988).
9. M. Combescure, *Ann. Inst. Henri Poincaré* **47**:63 (1987).
10. M. Combescure, *Ann. Phys.* **185**:86 (1988).
11. M. Combescure, *J. Stat. Phys.* **59** (1990).
12. B. Eckardt, *Phys. Rep.* **163**:3590 (1988).
13. S. Fishman, D. R. Grempel, and R. E. Prange, *Phys. Rev. A* **36**:289 (1987).
14. T. Geisel, *Phys. Rev. A* **41**:2989 (1990).
15. I. Guarneri, *Lett. Nuovo Cimento* **40**:171 (1984).
16. F. Haake, M. Kus, and R. Scharf, *Z. Phys. B* **65**:381 (1987).
17. J. Howland, *Indiana Math. J.* **28**:471 (1979).
18. J. Howland, *Ann. Inst. H. Poincaré A* **50**:309 (1989).
19. J. Howland, *Ann. Inst. H. Poincaré A* **50**:325 (1989).
20. S. Kotani, *Rev. Math. Phys.* **1**:129 (1989).
21. J. M. Luck, H. Orland, and U. Smilansky, *J. Stat. Phys.* **53**:551 (1988).
22. J. M. Luck, *Phys. Rev. B* **39**:5834 (1989).
23. B. Milek and P. Seba, Singular continuous quasi energy spectrum in the kicked rotator with separable perturbation: Onset of quantum chaos, Preprint, Ruhr Universität Bochum (1989).
24. J. Peyrière, private communication.
25. J. Piepenbrink, private communication.
26. M. Queffelec, *Substitution Dynamical Systems; Spectral Analysis* (Springer, Berlin, 1987).
27. E. Shimshoni and U. Smilansky, *Nonlinearly* **1**:435 (1988).
28. Y. Sinai, Mathematical problems in the theory of quantum chaos, Preprint, Moscow (1989).
29. B. Sutherland, *Phys. Rev. Lett.* **57**:770 (1986).
30. M. Kolar, M. K. Ali, and F. Nori, Generalized Thue–Morse chains and their properties, *Phys. Rev. B*, to appear.